

ON A COMPLETED GENERATING FUNCTION OF LOCALLY HARMONIC MAASS FORMS

KATHRIN BRINGMANN, BEN KANE, AND SANDER ZWEGERS

ABSTRACT. While investigating the Doi–Naganuma lift, Zagier defined integral weight cusp forms f_D which are naturally defined in terms of binary quadratic forms of discriminant D . It was later determined by Kohnen and Zagier that the generating function for the f_D is a half-integral weight cusp form. A natural preimage of f_D under a differential operator at the heart of the theory of harmonic weak Maass forms was determined by the first two authors and Kohnen. In this paper, we consider the modularity properties of the generating function of these preimages. We prove that although the generating function is not itself modular, it can be naturally completed to obtain a half-integral weight modular object.

1. INTRODUCTION AND STATEMENT OF RESULTS

Throughout we let $k \geq 4$ be an even integer. While investigating the Doi–Naganuma lift [12], Zagier [20] defined for $D > 0$ the function

$$f_D(\tau) := f_{k,D}(\tau) := D^{k-\frac{1}{2}} \sum_{Q \in \mathcal{Q}_D} \frac{1}{Q(\tau, 1)^k} \quad (\tau \in \mathbb{H}),$$

where \mathcal{Q}_D is the set of integral binary quadratic forms

$$[a, b, c](X, Y) := aX^2 + bXY + cY^2$$

of discriminant $D \in \mathbb{Z}$. Note that f_D has been renormalized from Zagier’s original definition. The function f_D is a weight $2k$ cusp form, while the generating function

$$\Omega(\tau, z) := \sum_{D > 0} f_D(\tau) e^{2\pi i D z} \quad (z \in \mathbb{H})$$

for the f_D is a modular form of weight $k + \frac{1}{2}$ in the z variable [14]. As was shown by the first two authors and Kohnen in [3], the functions f_D have natural weight $2 - 2k$ preimages $\mathcal{F}_D = \mathcal{F}_{1-k,D}$ under the operator $\xi_{2-2k} := 2iy^{2-2k} \frac{\partial}{\partial \bar{\tau}}$, which is central in the theory of

Date: March 7, 2013.

2010 Mathematics Subject Classification. 11F27, 11F25, 11F37, 11E16, 11F11.

Key words and phrases. locally harmonic Maass forms, indefinite theta functions, holomorphic projection, mock modular forms, modular forms.

The research of the first author was supported by the Alfried Krupp Prize for Young University Teachers of the Krupp Foundation.

harmonic weak Maass forms. In this paper we investigate the modularity properties of the generating function

$$\Psi(\tau, z) := \sum_{D>0} \mathcal{F}_D(\tau) e^{2\pi i D z}.$$

However, unlike in the case of Ω , Ψ is not itself modular, but may be naturally completed to a function which is modular of weight $\frac{3}{2} - k$ as a function of z . This mirrors the mock theta functions of Ramanujan, which are themselves holomorphic but may be completed to nonholomorphic modular objects called harmonic weak Maass forms. The mock theta functions are in a class of functions called *mock modular forms*, which have naturally appeared in a variety of applications. Their benefit has been observed in the areas of partition theory (for example [1, 2, 4, 5, 7]), Zagier's duality [23] (for example [6]), and derivatives of L -functions (for example [10, 11]). To give another example, they have also recently appeared in Eguchi, Ooguri, and Tachikawa's [13] investigation of moonshine for the largest Mathieu group M_{24} . For a good overview of mock modular forms, see [16] and [25].

We now return to the properties of the functions \mathcal{F}_D . In addition to being natural preimages of the functions f_D , the \mathcal{F}_D are furthermore *locally harmonic Maass forms*. Such functions satisfy weight $2 - 2k$ modularity and are annihilated (away from a certain set of measure zero) by the weight $2 - 2k$ hyperbolic Laplacian

$$\Delta_{2-2k} := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + i(2 - 2k)y \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) \quad (\tau = x + iy).$$

Denoting for $Q = [a, b, c] \in \mathcal{Q}_D$

$$Q_\tau := \frac{1}{y} (a|\tau|^2 + bx + c),$$

the functions \mathcal{F}_D are explicitly defined by

$$\mathcal{F}_D(\tau) := \frac{2}{\beta(k - \frac{1}{2}, \frac{1}{2})} \sum_{Q \in \mathcal{Q}_D} \text{sgn}(Q_\tau) Q(\tau, 1)^{k-1} \psi_k \left(\frac{Dy^2}{|Q(\tau, 1)|^2} \right).$$

Here

$$\psi_k(v) := \frac{1}{2} \beta \left(v; k - \frac{1}{2}, \frac{1}{2} \right)$$

is a special value of the incomplete β -function, which is given for $s, w \in \mathbb{C}$ satisfying $\text{Re}(s), \text{Re}(w) > 0$ by

$$\beta(v; s, w) := \int_0^v t^{s-1} (1-t)^{w-1} dt.$$

Moreover, for $\text{Re}(s), \text{Re}(w) > 0$, we denote $\beta(s, w) := \beta(1; s, w)$. Note that we have renormalized the definition of \mathcal{F}_D given in [3].

To complete Ψ , we define for $D \in \mathbb{Z}$

$$(1.1) \quad \mathcal{G}_D(v; \tau) := -\frac{1}{\sqrt{\pi}} \sum_{Q \in \mathcal{Q}_D} \text{sgn}(Q_\tau) Q(\tau, 1)^{k-1} \Gamma \left(\frac{1}{2}; 4\pi Q_\tau^2 v \right) \quad (z = u + iv),$$

where

$$\Gamma(s; w) := \int_w^\infty t^{s-1} e^{-t} dt \quad (w > 0, s \in \mathbb{C})$$

is the incomplete gamma function. We denote the generating function for the \mathcal{G}_D by

$$\Psi^*(\tau, z) := \sum_{D \in \mathbb{Z}} \mathcal{G}_D(v; \tau) e^{2\pi i D z}.$$

We then define the completion of Ψ by

$$(1.2) \quad \widehat{\Psi}(\tau, z) := \Psi(\tau, z) + \Psi^*(\tau, z).$$

To state the modularity properties of $\widehat{\Psi}$, we set for $\kappa \in \frac{1}{2}\mathbb{Z}$

$$\Gamma := \begin{cases} \mathrm{SL}_2(\mathbb{Z}) & \text{if } \kappa \in \mathbb{Z}, \\ \Gamma_0(4) & \text{if } \kappa \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}. \end{cases}$$

Let \mathfrak{M}_κ denote the space of real analytic functions $f : \mathbb{H} \rightarrow \mathbb{C}$ satisfying weight $\kappa \in \frac{1}{2}\mathbb{Z}$ modularity for Γ , with the additional restriction that f is in Kohnen's plus space whenever $\kappa \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$. For a formal definition of Kohnen's plus space, see the comments preceding Lemma 2.2.

Theorem 1.1. *As a function of z , $\widehat{\Psi}(\tau, z)$ is an element of $\mathfrak{M}_{k+\frac{1}{2}}$, while as a function of τ it is an element of \mathfrak{M}_{2-2k} .*

Remarks.

- (1) One can show that, as a function of z , $\widehat{\Psi}$ satisfies the growth conditions of a cusp form, i.e., $v^{\frac{k}{2}+\frac{1}{4}}|\widehat{\Psi}(\tau, z)|$ is bounded on \mathbb{H} . However, the corresponding growth condition in τ is not satisfied by $\widehat{\Psi}$.
- (2) Since the functions \mathcal{F}_D exhibit discontinuities along certain geodesics, it is somewhat surprising that $\widehat{\Psi}$ is real analytic in τ .

The function $\widehat{\Psi}$ is furthermore naturally connected to Ω and indefinite theta functions through the weight lowering operator $L_w := \mathrm{Im}(w)^2 \frac{\partial}{\partial \bar{w}}$, which sends functions satisfying weight κ modularity to functions which satisfy weight $\kappa - 2$ modularity. The theta functions we require are

$$\Theta_1(\tau, z) := iv^{\frac{3}{2}} \sum_{\substack{D \in \mathbb{Z} \\ Q \in \mathcal{Q}_D}} Q(\tau, 1)^{k-1} Q_\tau e^{-4\pi Q_\tau^2 v} e^{2\pi i D z}$$

and (the projection into Kohnen's plus space of) Shintani's [17] classical (non-holomorphic) theta kernel

$$\Theta_2(\tau, z) := 2iv^{\frac{1}{2}} y^{-2k} \sum_{\substack{D \in \mathbb{Z} \\ Q \in \mathcal{Q}_D}} Q(\tau, 1)^k e^{-4\pi Q_\tau^2 v} e^{2\pi i D z}.$$

Although these two theta functions have known modularity properties, we supply direct proofs of this modularity in Lemma 2.2 and Lemma 2.3 as a convenience to the reader. Specifically, the function $\Theta_1(\tau, z)$ is a weight $k - \frac{3}{2}$ indefinite theta function for $\Gamma_0(4)$ in

Kohnen's plus space in z and satisfies weight $2 - 2k$ modularity for $\mathrm{SL}_2(\mathbb{Z})$ in τ . The function $\Theta_2(-\bar{\tau}, z)$ is a weight $k + \frac{1}{2}$ indefinite theta function for $\Gamma_0(4)$ in Kohnen's plus space in z and satisfies weight $2k$ modularity for $\mathrm{SL}_2(\mathbb{Z})$ in τ .

Theorem 1.2.

(1) *The image of the function $\widehat{\Psi}$ under the lowering operator in z equals*

$$(1.3) \quad L_z \left(\widehat{\Psi}(\tau, z) \right) = \Theta_1(\tau, z).$$

(2) *The image of the function $\widehat{\Psi}$ under the lowering operator in τ equals*

$$(1.4) \quad y^{-2k} L_\tau \left(\widehat{\Psi}(\tau, z) \right) = \frac{i}{\beta \left(k - \frac{1}{2}, \frac{1}{2} \right)} \Omega_k(-\bar{\tau}, z) - \Theta_2(\tau, z).$$

Remark. Bruinier, Funke, and Imamoglu communicated to us that they obtained analogous results to our Theorems 1.1 and 1.2 for the case $k = 0$ [9]. Their approach is based on extending the theta lift considered in [8] to meromorphic modular functions.

The paper is organized as follows. In Section 2 we use a theorem of Vignéras [19] to supply a direct proof of the modularity of Θ_1 and Θ_2 . Section 3 is devoted to holomorphic projection, which is a key ingredient in the proof of Theorem 1.1. Section 4 is centered around the convergence of $\widehat{\Psi}$ and its real analyticity. The modularity of $\widehat{\Psi}$ is established in Section 5.

2. INDEFINITE THETA FUNCTIONS

For $\kappa \in \frac{1}{2}\mathbb{Z}$, a finite index subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$, and a character χ , we say that a function $f : \mathbb{H} \rightarrow \mathbb{C}$ is *modular of weight κ for Γ with character χ* if for every $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma$ one has $f|_\kappa M = \chi(\delta) f$. Here $|_\kappa$ is the usual weight κ slash operator.

To show the modularity of the indefinite theta functions which we encounter in this paper, we will employ a result of Vignéras [19]. For this, we define the Euler operator $E := \sum_{i=1}^n w_i \frac{\partial}{\partial w_i}$. As usual, we denote the Gram matrix associated to a nondegenerate quadratic form q on \mathbb{R}^n by A . The *Laplacian* associated to q is then defined by $\Delta := \left\langle \frac{\partial}{\partial w}, A^{-1} \frac{\partial}{\partial w} \right\rangle$. Here $\langle \cdot, \cdot \rangle$ denotes the usual inner product on \mathbb{R}^n .

Theorem 2.1 (Vignéras). *Suppose that $n \in \mathbb{N}$, q is a nondegenerate quadratic form on \mathbb{R}^n , $L \subset \mathbb{R}^n$ is a lattice on which q takes integer values, and $p : \mathbb{R}^n \rightarrow \mathbb{C}$ is a function satisfying the following conditions:*

- (i) *The function $f(w) := p(w)e^{-2\pi q(w)}$ times any polynomial of degree at most 2 and all partial derivatives of f of order at most 2 are elements of $L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$.*
- (ii) *For some $\lambda \in \mathbb{Z}$, the function p satisfies*

$$\left(E - \frac{\Delta}{4\pi} \right) p = \lambda p.$$

Then the indefinite theta function

$$v^{-\frac{\lambda}{2}} \sum_{w \in L} p(w\sqrt{v}) e^{2\pi i q(w)z}$$

is modular of weight $\lambda + \frac{n}{2}$ for $\Gamma_0(N)$ and character χ , where N and χ are the level and character of q .

We use Theorem 2.1 to show the modularity of the theta functions Θ_1 and Θ_2 . To state the modularity, recall that \mathfrak{M}_κ denotes the space of real analytic functions $f : \mathbb{H} \rightarrow \mathbb{C}$ satisfying weight $\kappa \in \frac{1}{2}\mathbb{Z}$ modularity for Γ , with the additional restriction that f is in Kohnen's plus space whenever $\kappa \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$. Here we say that a function satisfying weight $\ell + \frac{1}{2}$ modularity is an element of Kohnen's plus space if its Fourier expansion has the shape

$$\sum_{(-1)^\ell n \equiv 0,1 \pmod{4}} a_n(v) e^{2\pi i n z}.$$

Lemma 2.2. *As a function of z , $\Theta_1(\tau, z) \in \mathfrak{M}_{k-\frac{3}{2}}$. Furthermore, as a function of τ , $\Theta_1(\tau, z) \in \mathfrak{M}_{2-2k}$.*

Proof. To prove the modularity in z , we use Theorem 2.1 with $q(a, b, c) = b^2 - 4ac$, $L = \mathbb{Z}^3$, and

$$p(a, b, c) = Q_\tau Q(\tau, 1)^{k-1} e^{-4\pi Q_\tau^2}.$$

One sees directly that

$$p(\sqrt{v}a, \sqrt{v}b, \sqrt{v}c) = v^{\frac{k}{2}} Q_\tau Q(\tau, 1)^{k-1} e^{-4\pi Q_\tau^2 v}.$$

We next note that

$$(2.1) \quad |Q(\tau, 1)|^2 = Q_\tau^2 y^2 + D y^2.$$

It is then straightforward to show that

$$(2.2) \quad D + 2Q_\tau^2 = \frac{2}{y^2} |Q(\tau, 1)|^2 - D$$

is positive definite. From this, one can easily verify that condition (i) of Theorem 2.1 is satisfied.

A straightforward calculation yields

$$E(p(a, b, c)) = (k - 8\pi Q_\tau^2) p(a, b, c)$$

and

$$\Delta(p(a, b, c)) = 4\pi (3 - 8\pi Q_\tau^2) p(a, b, c).$$

Thus $\lambda = k - 3$ in Theorem 2.1. This gives that $\Theta_1(\tau, z)$ is an indefinite theta function of weight $k - \frac{3}{2}$ for $\Gamma_0(4)$. Since $k - 2$ is even, one sees that the plus space condition is clearly satisfied for Θ_1 .

To prove the modularity in the τ variable, we directly apply translation and inversion. By making the change of variables $b \rightarrow b + 2a$ and $c \rightarrow a + b + c$, one sees by term by term comparison that

$$\Theta_1(\tau + 1, z) = \Theta_1(\tau, z).$$

Similarly, the change of variables $a \rightarrow c$, $b \rightarrow -b$, and $c \rightarrow a$ implies that

$$\Theta_1\left(-\frac{1}{\tau}, z\right) = \tau^{2-2k} \Theta_1(\tau, z).$$

□

We next recall the modularity properties of Θ_2 .

Lemma 2.3. *As a function of z , $\Theta_2(-\bar{\tau}, z) \in \mathfrak{M}_{k+\frac{1}{2}}$. As a function of τ , $\Theta_2(-\bar{\tau}, z) \in \mathfrak{M}_{2k}$.*

Proof. The proof follows by the same argument as in Lemma 2.2, using Theorem 2.1 with $q(a, b, c) = b^2 - 4ac$, $L = \mathbb{Z}^3$, and

$$p(a, b, c) = Q(\tau, 1)^k e^{-4\pi Q_\tau^2}.$$

Here, a straightforward calculation leads to

$$E(p(a, b, c)) = (k - 8\pi Q_\tau^2) p(a, b, c)$$

and

$$\Delta(p(a, b, c)) = 4\pi (1 - 8\pi Q_\tau^2) p(a, b, c),$$

from which one concludes that $\lambda = k - 1$, and the modularity in z follows.

The modularity in τ follows by the same changes of variables given in the proof of Lemma 2.2.

□

3. HOLOMORPHIC PROJECTION

In this section we introduce the holomorphic projection operator and investigate some of its basic properties. In the integer weight case, these properties were first proven by Sturm [18] and a good overview may be found in Appendix C of [22]. For a translation invariant function $f : \mathbb{H} \rightarrow \mathbb{C}$ we write its Fourier expansion as

$$(3.1) \quad f(z) = \sum_{r \in \mathbb{Z}} c_r(v) e^{2\pi i r z}.$$

We formally define the *weight κ holomorphic projection of f* by

$$\pi_\kappa(f)(z) := \pi_{\kappa, z}(f)(z) := \sum_{r \in \mathbb{N}} c_r e^{2\pi i r z},$$

where

$$(3.2) \quad c_r := \frac{(4\pi r)^{\kappa-1}}{\Gamma(\kappa-1)} \int_0^\infty c_r(t) e^{-4\pi r t} t^{\kappa-2} dt.$$

Here $\Gamma(s)$ is the usual gamma function.

Lemma 3.1. *If $f : \mathbb{H} \rightarrow \mathbb{C}$ is a translation invariant function, then*

$$(3.3) \quad \pi_\kappa(f)(z) = \frac{(\kappa - 1)(2i)^\kappa}{4\pi} \int_{\mathbb{H}} \frac{f(\tau)y^\kappa}{(z - \bar{\tau})^\kappa} \frac{dx dy}{y^2},$$

for every $1 < \kappa \in \frac{1}{2}\mathbb{Z}$ for which the right hand side of (3.3) converges absolutely.

Remark. In the case that $\kappa \in \mathbb{Z}$, the integral in Lemma 3.1 appears in the proof of the trace formula for the Hecke operators established in [21].

Proof. Using the fact that f is translation invariant, we rewrite the integral on the right hand side of (3.3) as

$$\int_0^\infty y^{\kappa-2} \int_0^1 f(x + iy) \sum_{n \in \mathbb{Z}} \frac{1}{(z - x + iy + n)^\kappa} dx dy.$$

After inserting the Fourier expansion of f , the result follows by a special case of the Lipschitz summation formula [15], which yields

$$\sum_{n \in \mathbb{Z}} \frac{1}{(w + n)^\kappa} = \frac{(-2\pi i)^\kappa}{\Gamma(\kappa)} \sum_{n \in \mathbb{N}} n^{\kappa-1} e^{2\pi i n w} \quad (w \in \mathbb{H}).$$

□

An easy change of variables in (3.3) immediately implies that holomorphic projection commutes with the weight κ slash operator.

Lemma 3.2. *If the right hand side of (3.3) converges absolutely for $\kappa \in \frac{1}{2}\mathbb{Z}$, then one has for every $M \in \mathrm{SL}_2(\mathbb{Z})$*

$$\pi_\kappa(f)|_\kappa M = \pi_\kappa(f|_\kappa M).$$

Combining Lemmas 3.1 and 3.2 yields the following special case.

Lemma 3.3. *If $|f(z)|v^r$ is bounded on \mathbb{H} and $\kappa \in \frac{1}{2}\mathbb{Z}$ satisfies $\kappa > r + 1 > 1$, then for every $M \in \mathrm{SL}_2(\mathbb{Z})$ one has*

$$(3.4) \quad \pi_\kappa(f)|_\kappa M = \pi_\kappa(f|_\kappa M).$$

Moreover, $|\pi_\kappa(f)(z)|v^r$ is bounded on \mathbb{H} .

Proof. Making the change of variables $x \rightarrow x(y + v) + u$ and then $y \rightarrow yv$, we may bound the integral of the absolute value by

$$(3.5) \quad \int_0^\infty \frac{y^{\kappa-2}}{(1+y)^{\kappa-1}} \int_{-\infty}^\infty \frac{|f(xv(1+y) + u + ivy)|}{(x^2 + 1)^{\frac{\kappa}{2}}} dx dy \\ \ll v^{-r} \int_0^\infty \frac{y^{\kappa-r-2}}{(1+y)^{\kappa-1}} dy \int_0^\infty \frac{1}{(x^2 + 1)^{\frac{\kappa}{2}}} dx.$$

Here we have used the assumed bound for f . The integral over x converges for $\kappa > 1$ and the integral over y converges for $\kappa > r + 1 > 1$. Lemma 3.2 now yields (3.4) while (3.5) further implies that $|\pi_\kappa(f)(z)|v^r$ is bounded on \mathbb{H} . □

Using the integral representation of the gamma function, one obtains that the holomorphic projection operator acts trivially on holomorphic functions.

Lemma 3.4. *If $f : \mathbb{H} \rightarrow \mathbb{C}$ is holomorphic and translation invariant, then for every $\kappa \in \frac{1}{2}\mathbb{Z}$ for which (3.2) converges, one has*

$$\pi_\kappa(f) = f.$$

4. CONVERGENCE AND SINGULARITIES OF $\widehat{\Psi}$

We first prove absolute convergence of $\widehat{\Psi}$.

Proposition 4.1. *The sums defining the two summands Ψ and Ψ^* of $\widehat{\Psi}$ in (1.2) converge absolutely.*

Proof. By (4.6) and (4.11) of [3], one easily deduces that \mathcal{F}_D converges absolutely and grows at most polynomially as a function of D . Therefore Ψ converges absolutely.

We next show that

$$(4.1) \quad \mathcal{G}_D(v; \tau) \ll_{v, \tau} D^{\frac{3k}{2}} \cdot \begin{cases} 1 & \text{if } D \geq 0, \\ e^{-4\pi|D|v} & \text{if } D < 0. \end{cases}$$

For ease of notation, we abbreviate

$$\alpha_D = \alpha_{D, v} := \begin{cases} 1 & \text{if } D \geq 0, \\ e^{-4\pi|D|v} & \text{if } D < 0. \end{cases}$$

We make frequent use of the bound

$$(4.2) \quad \Gamma\left(\frac{1}{2}; r\right) \ll e^{-r} \quad (r \geq 0).$$

Here and throughout the implied constant only depends on k unless otherwise noted.

We first consider the case when $a \neq 0$ and $D \neq 0$ and assume without loss of generality that $a > 0$ by the change of variables $(a, b, c) \rightarrow (-a, -b, -c)$. We then rewrite b as $b + 2an$ with $0 \leq b < 2a$ and $n \in \mathbb{Z}$ and split the sum into those summands with $|n|$ “large” and those with $|n|$ “small.”

We begin with the case $|n| > 8(|\tau| + \sqrt{|D|})$. By a straightforward generalization of (4.4) of [3] (with $D \rightarrow |D|$), we have

$$(4.3) \quad |Q_\tau| \gg \frac{an^2}{y}.$$

Noting that we trivially bounded $|x| < |\tau|$ to obtain (4.4) of [3], one also immediately obtains $|Q(\tau, 1)| \gg an^2$. Thus (4.3) of [3] implies that

$$(4.4) \quad |Q(\tau, 1)| \asymp an^2.$$

For $D < 0$, (2.1) combined with (4.4) yields the further bound

$$Q_\tau^2 \gg \frac{a^2 n^4}{y^2} + |D|.$$

Combining this with (4.2) and (4.3), we obtain

$$(4.5) \quad \Gamma\left(\frac{1}{2}; 4\pi Q_\tau^2 v\right) \ll \alpha_D e^{-\frac{4\pi a^2 n^4 v}{y^2}} \ll \alpha_D e^{-4\pi a^2 n^2 v},$$

where we have used the fact that $n^2 > y^2$. Thus (4.5) and (4.4) yield the bound

$$\begin{aligned} \sum_{a \in \mathbb{N}} \sum_{\substack{0 \leq b < 2a \\ b^2 \equiv D \pmod{4a}}} \sum_{|n| > 8(|\tau| + \sqrt{|D|})} \left| Q(\tau, 1)^{k-1} \Gamma\left(\frac{1}{2}; 4\pi Q_\tau^2 v\right) \right| \\ \ll \alpha_D \sum_{a \in \mathbb{N}} \sum_{\substack{0 \leq b < 2a \\ b^2 \equiv D \pmod{4a}}} \sum_{|n| > 8(|\tau| + \sqrt{|D|})} a^{k-1} n^{2k-2} e^{-4\pi a^2 n^2 v} \\ \ll \alpha_D \sum_{a \in \mathbb{N}} a^k \sum_{n \in \mathbb{N}} n^{2k-2} e^{-4\pi a^2 n^2 v} \ll_v \alpha_D. \end{aligned}$$

We next consider the case $|n| \leq 8(|\tau| + \sqrt{|D|})$. By (4.7) of [3], one has

$$(4.6) \quad |Q(\tau, 1)| \ll a \left(|\tau| + \sqrt{|D|} \right)^2.$$

For $a > \frac{\sqrt{|D|}}{y}$, (4.8) of [3] implies that

$$|Q_\tau| \gg ay, \quad |Q(\tau, 1)| \gg ay^2.$$

Thus by (4.2) we have

$$(4.7) \quad \Gamma\left(\frac{1}{2}; 4\pi Q_\tau^2 v\right) \ll \alpha_D e^{-4\pi a^2 y^2 v}.$$

Hence, combining (4.7) with (4.6) gives that

$$\begin{aligned} \sum_{a > \frac{\sqrt{|D|}}{y}} \sum_{\substack{0 \leq b < 2a \\ b^2 \equiv D \pmod{4a}}} \sum_{|n| \leq 8(|\tau| + \sqrt{|D|})} \left| Q(\tau, 1)^{k-1} \Gamma\left(\frac{1}{2}; 4\pi Q_\tau^2 v\right) \right| \\ \ll \alpha_D \left(|\tau| + \sqrt{|D|} \right)^{2k-1} \sum_{a > \frac{\sqrt{|D|}}{y}} a^k e^{-4\pi a^2 y^2 v} \\ \ll \alpha_D \left(|\tau| + \sqrt{|D|} \right)^{2k-1} \sum_{a \in \mathbb{N}} a^k e^{-4\pi a^2 y^2 v} \ll_{\tau, v} \alpha_D |D|^{k-\frac{1}{2}}. \end{aligned}$$

For $a \leq \frac{\sqrt{|D|}}{y}$, we trivially bound

$$\Gamma\left(\frac{1}{2}; 4\pi Q_\tau^2 v\right) \ll \alpha_D.$$

Hence by (4.6), we have

$$\begin{aligned}
(4.8) \quad & \sum_{a \leq \frac{\sqrt{|D|}}{y}} \sum_{\substack{0 \leq b < 2a \\ b^2 \equiv D \pmod{4a}}} \sum_{|n| \leq 8(|\tau| + \sqrt{|D|})} \left| Q(\tau, 1)^{k-1} \Gamma\left(\frac{1}{2}; 4\pi Q_\tau^2 v\right) \right| \\
& \ll \alpha_D \left(|\tau| + \sqrt{|D|} \right)^{2k-2} \sum_{a \leq \frac{\sqrt{|D|}}{y}} \sum_{|n| \leq 8(|\tau| + \sqrt{|D|})} a^k \\
& \ll \alpha_D \left(|\tau| + \sqrt{|D|} \right)^{2k-1} \frac{|D|^{\frac{k+1}{2}}}{y^{k+1}} \ll_{\tau} \alpha_D |D|^{\frac{3k}{2}}.
\end{aligned}$$

If $a = 0$ and $D \neq 0$, then we reverse the roles of a and c in the above calculations, with $-\frac{1}{\tau}$ taking the place of τ .

We finally consider the case $D = 0$. Since $b^2 = 4ac$ we may assume without loss of generality that $a \geq 0$ and $c > 0$. We rewrite

$$Q(\tau, 1) = (\sqrt{a}\tau \pm \sqrt{c})^2$$

and

$$(4.9) \quad Q_\tau = \frac{1}{y} |\sqrt{a}\tau \pm \sqrt{c}|^2 \geq ay.$$

Thus we may rewrite

$$\begin{aligned}
(4.10) \quad & \sum_{a \in \mathbb{N}_0} \sum_{c \in \mathbb{N}} |Q(\tau, 1)|^{k-1} \Gamma\left(\frac{1}{2}; 4\pi Q_\tau^2 v\right) \\
& = \sum_{a \in \mathbb{N}_0} \sum_{c \in \mathbb{N}} \sum_{\pm} |\sqrt{a}\tau \pm \sqrt{c}|^{2k-2} \Gamma\left(\frac{1}{2}; \frac{4\pi v}{y^2} |\sqrt{a}\tau \pm \sqrt{c}|^4\right).
\end{aligned}$$

For $a = 0$ we have $yQ_\tau = Q(\tau, 1) = c$, so that by (4.2) the resulting contribution to (4.10) may be bounded by

$$\sum_{c \in \mathbb{N}} c^{k-1} e^{-\frac{4\pi v c^2}{y^2}} \ll_{\tau, v} 1.$$

For $0 < c \leq 4a|\tau|^2$ we estimate

$$|Q(\tau, 1)| \ll a|\tau|^2.$$

Using (4.9) in (4.2), the contribution to (4.10) from the terms with $0 < c \leq 4a|\tau|^2$ is thus bounded by

$$|\tau|^{2k-2} \sum_{a \in \mathbb{N}} \sum_{0 < c \leq 4a|\tau|^2} a^{k-1} e^{-4\pi a^2 y^2 v} \ll |\tau|^{2k} \sum_{a \in \mathbb{N}} a^k e^{-4\pi a^2 y^2 v} \ll_{\tau, v} 1.$$

Finally, when $a \neq 0$ and $c > 4a|\tau|^2$, one has

$$|Q(\tau, 1)| \ll c$$

and

$$|Q_\tau| \geq \frac{c}{4y} \geq \frac{a|\tau|^2}{y}.$$

Therefore, the contribution to (4.10) from $c > 4a|\tau|^2$ and $a \neq 0$ is, up to a constant, bounded by

$$\sum_{a \in \mathbb{N}} \sum_{c > 4a|\tau|^2} c^{k-1} e^{-\frac{\pi a c |\tau|^2 v}{y^2}} \ll \sum_{a \in \mathbb{N}} e^{-\frac{\pi a |\tau|^2 v}{2y^2}} \sum_{c \in \mathbb{N}} c^{k-1} e^{-\frac{\pi c |\tau|^2 v}{2y^2}} \ll_{\tau, v} 1,$$

where we have used the fact that $ac \geq \frac{a+c}{2}$ for all $a, c \in \mathbb{N}$. Hence \mathcal{G}_0 converges absolutely. \square

We next rewrite $\widehat{\Psi}$ in terms of other special functions. In order to do so, we fix $\tau_0 = x_0 + iy_0 \in \mathbb{H}$. For $r \in \mathbb{R}$ we use the Gauss error function

$$\operatorname{erf}(r) := \frac{2}{\sqrt{\pi}} \int_0^r e^{-t^2} dt$$

to define

$$g_k(r) := \frac{1}{\Gamma(k - \frac{1}{2})} \int_0^\infty \operatorname{erf}\left(rt^{\frac{1}{2}}\right) e^{-t} t^{k-\frac{3}{2}} dt.$$

We furthermore formally define

$$(4.11) \quad \Psi_1(\tau, z) := - \sum_{\substack{D \in \mathbb{Z} \\ Q \in \mathcal{Q}_D}} Q(\tau, 1)^{k-1} \left(\operatorname{sgn}(Q_{\tau_0}) - \operatorname{erf}(2Q_\tau \sqrt{\pi v}) \right) e^{2\pi i D z}$$

and

$$(4.12) \quad \Psi_2(\tau, z) := \sum_{\substack{D > 0 \\ Q \in \mathcal{Q}_D}} Q(\tau, 1)^{k-1} \left(\operatorname{sgn}(Q_{\tau_0}) - g_k\left(\frac{Q_\tau}{\sqrt{D}}\right) \right) e^{2\pi i D z}.$$

We then rewrite $\widehat{\Psi}$ in the following lemma.

Lemma 4.2. *The sums Ψ_1 and Ψ_2 are absolutely convergent and*

$$(4.13) \quad \widehat{\Psi} = \Psi_1 + \Psi_2.$$

Before proving Lemma 4.2, we first rewrite g_k .

Lemma 4.3. *We have*

$$g_k(r) = \operatorname{sgn}(r) - \frac{2}{\beta\left(k - \frac{1}{2}, \frac{1}{2}\right)} \operatorname{sgn}(r) \psi_k\left(\frac{1}{1+r^2}\right).$$

Proof. Using the fact that

$$\beta(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)},$$

we compute

$$(4.14) \quad g'_k(r) = \frac{2}{\beta\left(k - \frac{1}{2}, \frac{1}{2}\right) (1+r^2)^k}.$$

Moreover $g_k(0) = 0$ since $\operatorname{erf}(0) = 0$. Thus

$$g_k(r) = \frac{2 \operatorname{sgn}(r)}{\beta\left(k - \frac{1}{2}, \frac{1}{2}\right)} \int_0^{|r|} \frac{1}{(1+t^2)^k} dt.$$

Making the change of variables $t \rightarrow \sqrt{\frac{1}{t} - 1}$ easily gives the claim of the lemma. \square

Proof of Lemma 4.2. First recall that

$$(4.15) \quad \operatorname{erf}(\sqrt{\pi}t) = \operatorname{sgn}(t) \left(1 - \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{1}{2}; \pi t^2\right) \right).$$

By Proposition 4.1 the function $\widehat{\Psi}$ converges absolutely. We add

$$(4.16) \quad \Psi_3(\tau, z) := \sum_{\substack{D \in \mathbb{Z} \\ Q \in \mathcal{Q}_D}} (\operatorname{sgn}(Q_{\tau_0}) - \operatorname{sgn}(Q_{\tau})) Q(\tau, 1)^{k-1} e^{2\pi i D z}$$

to Ψ and subtract it from Ψ^* . We then compare the D -th Fourier coefficient (with respect to $e^{2\pi i u}$) on both sides of (4.13). Combining (4.15) and Lemma 4.3, it remains to show that Ψ_3 converges absolutely and that whenever $D \leq 0$

$$(4.17) \quad \operatorname{sgn}(Q_{\tau}) = \operatorname{sgn}(Q_{\tau_0}).$$

We first consider the case $D \neq 0$ and start with $a \neq 0$, which is trivially satisfied when $D < 0$. Since $a \neq 0$, we have

$$(4.18) \quad Q_{\tau} = \frac{a}{y} \left(y^2 + \left(x + \frac{b}{2a} \right)^2 - \frac{D}{4a^2} \right).$$

Hence (4.18) (used with both τ and τ_0) implies (4.17) for every $Q \in \mathcal{Q}_D$ with $D < 0$, since

$$\operatorname{sgn}(Q_{\tau}) = \operatorname{sgn}(a) = \operatorname{sgn}(Q_{\tau_0}).$$

We next consider the case when $D > 0$. Whenever $|a| > \frac{\sqrt{D}}{2y}$, one sees by (4.18) that

$$(4.19) \quad \operatorname{sgn}(Q_{\tau}) = \operatorname{sgn}(a).$$

Similarly, whenever

$$(4.20) \quad \left| x + \frac{b}{2a} \right| > \frac{\sqrt{D}}{2|a|}$$

one sees by (4.18) that (4.19) holds. Note that (4.20) is satisfied in particular for

$$|b| > 2|ax| + \sqrt{D}.$$

One concludes that the number of $[a, b, c] \in \mathcal{Q}_D$ ($D > 0$) with $a \neq 0$ for which (4.17) does not hold grows at most polynomially in D .

We finally consider the case $a = 0$. Note that

$$\operatorname{sgn}(Q_{\tau}) = \operatorname{sgn}(bx + c) = \operatorname{sgn} \left(-b \operatorname{Re} \left(-\frac{1}{\tau} \right) + c \left| -\frac{1}{\tau} \right|^2 \right) = \operatorname{sgn} \left(\tilde{Q}_{-\frac{1}{\tau}} \right),$$

where $\tilde{Q} = [c, -b, 0]$. There are at most two choices of b for which $[0, b, 0] \in \mathcal{Q}_D$. Hence by the change of variables $\tau \rightarrow -\frac{1}{\tau} \in \mathbb{H}$ and $c \rightarrow a$, the same argument used for $a \neq 0$ implies that the number of $[0, b, c] \in \mathcal{Q}_D$ for which (4.17) does not hold grows at most polynomially as a function of D .

If $D = 0$ and $a \neq 0$, then (4.18) implies that $\text{sgn}(Q_\tau) = \text{sgn}(Q_{\tau_0})$. However, if $D = 0$ and $a = 0$, then $yQ_\tau = c = Q(\tau, 1)$. We immediately obtain (4.17) for $D = 0$ and furthermore conclude that Ψ_3 converges absolutely. This implies the claim. \square

5. MODULARITY AND HOLOMORPHIC PROJECTION

In this section, we prove Theorem 1.1 and Theorem 1.2. A key step in determining modularity is to use holomorphic projection. In order to do so, we show that Ψ_1 satisfies the growth conditions necessary to apply Lemma 3.3. The following lemma proves useful for this purpose.

Lemma 5.1. *Suppose that Q^+ is a positive definite ternary quadratic form and $v > 0$. Then the sum*

$$(5.1) \quad v^{\frac{k}{2}+1} \sum_{a,b,c \in \mathbb{Z}} |a\tau^2 + b\tau + c|^{k-1} e^{-2\pi Q^+(a,b,c)v}$$

converges absolutely and is bounded as a function of v .

Proof. We first note that

$$|a\tau^2 + b\tau + c| \ll_\tau |a| + |b| + |c|.$$

Furthermore, since Q^+ is positive definite, there exists a constant $\delta > 0$ such that

$$2\pi Q^+(a, b, c) \geq \delta (a^2 + b^2 + c^2).$$

Therefore, (5.1) can be bounded by

$$v^{\frac{k}{2}+1} \sum_{a,b,c \in \mathbb{Z}} (|a| + |b| + |c|)^{k-1} e^{-\delta(a^2+b^2+c^2)v}.$$

By the binomial theorem, it suffices to bound sums of the type

$$\sum_{a,b,c \in \mathbb{N}_0} a^{\ell_1} b^{\ell_2} c^{\ell_3} e^{-\delta(a^2+b^2+c^2)v}$$

with $\ell_1 + \ell_2 + \ell_3 = k - 1$. Using Proposition 3 of [24], one obtains for $v \rightarrow 0$

$$\sum_{n \in \mathbb{N}} n^\ell e^{-\delta n^2 v} = v^{-\frac{\ell}{2}} \sum_{n \in \mathbb{N}} (n\sqrt{v})^\ell e^{-\delta(n\sqrt{v})^2} \sim v^{-\frac{1}{2}(\ell+1)} \int_0^\infty w^\ell e^{-\delta w^2} dw \ll v^{-\frac{1}{2}(\ell+1)}.$$

Combining the above bound with the obvious exponential decay of (5.1) as $v \rightarrow \infty$ then yields the claim of the lemma. \square

We next show that the growth conditions are satisfied to apply holomorphic projection to Ψ_1 .

Lemma 5.2. *The function $v^{\frac{k}{2}+1} |\Psi_1(\tau, z)|$ is bounded.*

Proof. Since $\Psi_1 = \Psi^* - \Psi_3$, it suffices to bound Ψ_3 and Ψ^* . We begin with Ψ_3 . In order to apply Lemma 5.1, we first rewrite (4.16) in the notation of Lemma 2.6 of [26]. We set $q(a, b, c) = b^2 - 4ac$, $c_1 = (-1, 2x, -|\tau|^2)$, and $c_2 = (-1, 2x_0, -|\tau_0|^2)$. Denoting the bilinear form associated to q by $B(u_1, u_2) = q(u_1 + u_2) - q(u_1) - q(u_2)$, one computes for $w = (a, b, c)$:

$$\begin{aligned} q(c_1) &= -4y^2 < 0, & q(c_2) &= -4y_0^2 < 0, \\ B(c_1, w) &= 4yQ_\tau, & B(c_2, w) &= 4y_0Q_{\tau_0}, \\ B(c_1, c_2) &= -4(|\tau_0|^2 - 2xx_0 + |\tau|^2) < 0. \end{aligned}$$

Hence by Lemma 2.6 and (2.13) of [26], there exists a positive definite quadratic form Q^+ for which

$$\sum_{a,b,c \in \mathbb{Z}} |Q(\tau, 1)|^{k-1} |\operatorname{sgn}(Q_{\tau_0}) - \operatorname{sgn}(Q_\tau)| e^{-2\pi q(a,b,c)v} \ll \sum_{a,b,c \in \mathbb{Z}} |Q(\tau, 1)|^{k-1} e^{-2\pi Q^+(a,b,c)v}.$$

Since Q^+ is positive definite, Lemma 5.1 implies that the above sum converges and may be estimated against a constant times $v^{-\frac{k}{2}-1}$.

We next bound Ψ^* . Using (4.2), we estimate

$$\sum_{a,b,c \in \mathbb{Z}} |Q(\tau, 1)|^{k-1} \Gamma\left(\frac{1}{2}; 4\pi Q_\tau^2 v\right) e^{-2\pi Dv} \ll \sum_{a,b,c \in \mathbb{Z}} |Q(\tau, 1)|^{k-1} e^{-2\pi v(D+2Q_\tau^2)}.$$

Since (2.2) is positive definite, Lemma 5.1 concludes the proof. \square

By Lemma 5.2, we may now apply holomorphic projection in z to Ψ_1 . If the dependence on τ is clear, then we suppress it in what follows. We write

$$\Psi_1(z) = \sum_{D \in \mathbb{Z}} c_D(v) e^{2\pi i D z},$$

where

$$c_D(v) := - \sum_{Q \in \mathcal{Q}_D} Q(\tau, 1)^{k-1} (\operatorname{sgn}(Q_{\tau_0}) - \operatorname{erf}(2Q_\tau \sqrt{\pi v})).$$

Lemma 5.3. *One has that*

$$(5.2) \quad \widehat{\Psi} = \Psi_1 - \pi_{k+\frac{1}{2}}(\Psi_1).$$

Thus in particular

$$\pi_{k+\frac{1}{2}}(\widehat{\Psi}) = 0.$$

Proof. By (4.13), the lemma is equivalent to the statement that

$$\pi_{k+\frac{1}{2}}(\Psi_1) = -\Psi_2.$$

By Lemma 5.2 and Lemma 3.3, we may apply holomorphic projection to Ψ_1 since $k > 3$. Using the definition (3.2) of holomorphic projection, we compute

$$\pi_{k+\frac{1}{2}}(\Psi_1)(z) = \sum_{D \in \mathbb{N}} c_D e^{2\pi i D z}$$

with

$$\begin{aligned} c_D &= \frac{(4\pi D)^{k-\frac{1}{2}}}{\Gamma(k-\frac{1}{2})} \int_0^\infty c_D(v) e^{-4\pi Dv} v^{k-\frac{1}{2}} \frac{dv}{v} \\ &= -\frac{(4\pi D)^{k-\frac{1}{2}}}{\Gamma(k-\frac{1}{2})} \sum_{Q \in \mathcal{Q}_D} Q(\tau, 1)^{k-1} \int_0^\infty (\operatorname{sgn}(Q_{\tau_0}) - \operatorname{erf}(2Q_\tau \sqrt{\pi v})) e^{-4\pi Dv} v^{k-\frac{1}{2}} \frac{dv}{v}. \end{aligned}$$

We consider both integrals separately. The first summand is evaluated immediately by using the integral representation of the gamma function and the result follows by the definition of g_k . \square

We now prove Theorem 1.2.

Proof of Theorem 1.2. Note that by Lemma 5.3

$$L_z(\widehat{\Psi}(\tau, z)) = L_z(\Psi_1(\tau, z)),$$

because $\pi_{k+\frac{1}{2}}(\Psi_1)$ is holomorphic as a function of z . Hence (1.3) follows directly by

$$\frac{d}{dr} \operatorname{erf}(r) = \frac{2}{\sqrt{\pi}} e^{-r^2}.$$

In order to prove (1.4), we use Lemma 4.2 and apply L_τ to Ψ_1 and Ψ_2 . Using the fact that

$$(5.3) \quad L_\tau(Q_\tau) = \frac{1}{2i} Q(\tau, 1),$$

one obtains

$$L_\tau(\Psi_1(\tau, z)) = -y^{2k} \Theta_2(\tau, z).$$

Using (4.14) and (5.3), a short calculation using (2.1) shows that

$$L_\tau(\Psi_2(\tau, z)) = \frac{iy^{2k}}{\beta(k-\frac{1}{2}, \frac{1}{2})} \Omega(-\bar{\tau}, z).$$

\square

We now use the modularity of Θ_1 proven in Lemma 2.2 to obtain Theorem 1.1.

Proof of Theorem 1.1. Since L_z commutes with the slash operator, (1.3) and Lemma 2.2 imply that for $M \in \Gamma_0(4)$

$$L_z\left(\widehat{\Psi}\Big|_{k+\frac{1}{2}} M - \widehat{\Psi}\right) = L_z\left(\widehat{\Psi}\right)\Big|_{k-\frac{3}{2}} M - L_z\left(\widehat{\Psi}\right) = \Theta_1\Big|_{k-\frac{3}{2}} M - \Theta_1 = 0.$$

Thus,

$$\widehat{\Psi}\Big|_{k+\frac{1}{2}} M - \widehat{\Psi}$$

is holomorphic. We next use Lemma 5.3 followed by Lemma 3.3 and then Lemma 3.4. This yields

$$0 = \pi_{k+\frac{1}{2}} \left(\widehat{\Psi} \right) \Big|_{k+\frac{1}{2}} M - \pi_{k+\frac{1}{2}} \left(\widehat{\Psi} \right) = \pi_{k+\frac{1}{2}} \left(\widehat{\Psi} \Big|_{k+\frac{1}{2}} M - \widehat{\Psi} \right) = \widehat{\Psi} \Big|_{k+\frac{1}{2}} M - \widehat{\Psi}.$$

This implies the modularity in z . Recalling that k is even, a direct inspection of the Fourier expansion yields that Kohnen's plus space condition is satisfied.

The modularity in τ follows by the same changes of variables given in the proof of Lemma 2.2. To complete the proof, we note that the function $\widehat{\Psi}$ is real analytic due to the definitions (4.11) and (4.12) in the representation (4.13). □

REFERENCES

- [1] G. Andrews, *Partitions, Durfee symbols, and the Atkin–Garvan moments of ranks*, Invent. Math. **169** (2007), 37–73.
- [2] K. Bringmann, *On the explicit construction of higher deformations of partition statistics*, Duke Math. J. **144** (2008), 195–233.
- [3] K. Bringmann, B. Kane, and W. Kohnen, *Locally harmonic Maass forms and rational period functions*, submitted for publication.
- [4] K. Bringmann, F. Garvan, and K. Mahlburg, *Partition statistics and quasiharmonic Maass forms*, Int. Math. Res. Not. **2009** (2009), 63–97.
- [5] K. Bringmann and K. Ono, *The $f(q)$ mock theta function conjecture and partition ranks*, Invent. Math. **165** (2006), 243–266.
- [6] K. Bringmann and K. Ono, *Arithmetic properties of coefficients of half-integral weight Maass–Poincaré series*, Math. Ann. **337** (2007), 591–612.
- [7] K. Bringmann and K. Ono, *Dyson's ranks and Maass forms*, Ann. of Math. **171** (2010), 419–449.
- [8] J. Bruinier, J. Funke, and Ö. Imamoglu, *Regularized theta liftings and periods of modular functions*, preprint.
- [9] J. Bruinier, J. Funke, and Ö. Imamoglu, *Rational period functions, singular automorphic forms, and theta lifts*, in preparation.
- [10] J. Bruinier and K. Ono, *Heegner divisors, L -functions, and Maass forms*, Ann. of Math. **172** (2010), 2135–2181.
- [11] J. Bruinier and T. Yang, *Faltings heights of CM cycles and derivatives of L -functions*, Invent. Math. **177** (2009), 631–681.
- [12] K. Doi and H. Naganuma, *On the algebraic curves uniformized by arithmetical automorphic functions*, Ann. of Math. **86** (1967), 449–460.
- [13] T. Eguchi, H. Ooguri, and Y. Tachikawa, *Notes on the K3 surface and the Mathieu group M_{24}* , Exper. Math. **20** (2011), 91–96.
- [14] W. Kohnen and D. Zagier, *Values of L -series of modular forms at the center of the critical strip* **64** (1981), 175–198.
- [15] R. Lipschitz, *Untersuchung der Eigenschaften einer Gattung von unendlichen Reihen*, J. Reine und Angew. Math **105** (1889), 127–156.
- [16] K. Ono, *Unearthing the visions of a master: harmonic Maass forms and number theory*, Proceedings of the 2008 Harvard–MIT Current Developments in Mathematics Conference, International Press, Somerville, MA (2009), 347–454.
- [17] T. Shintani, *On construction of holomorphic cusp forms of half integral weight*, Nagoya Math. J. **58** (1975), 83–126.
- [18] J. Sturm, *Projections of C^∞ automorphic forms*, Bull. Amer. Math. Soc. **2** (1980), 435–439.

- [19] M. Vignéras, *Séries theta des formes quadratiques indéfinies* in: Modular functions of one variable VI, Springer lecture notes **627** (1977), 227–239.
- [20] D. Zagier, *Modular forms associated to real quadratic fields*, Invent. Math. **30** (1975), 1–46.
- [21] D. Zagier, *The Eichler-Selberg trace formula on $SL_2(\mathbb{Z})$* , Appendix to S. Lang, Introduction to modular forms, Grundlehren d. math. Wiss. **222**, Springer-Verlag, Berlin (1976), 44–54.
- [22] D. Zagier, *Introduction to modular forms* in From number theory to Physics, Springer-Verlag (1992), 238–291.
- [23] D. Zagier, *Traces of singular moduli* in Motives, Polylogarithms and Hodge Theory, Part I, International Press Lecture Series (Eds. F. Bogomolov and L. Katzarkov), International Press (2002), 211–244.
- [24] D. Zagier, *The Mellin transform and other useful analytic techniques*, Appendix to E. Zeidler, Quantum Field Theory I: Basics in Mathematics and Physics. A Bridge Between Mathematicians and Physicists, Springer-Verlag, Berlin-Heidelberg-New York (2006), 305–323.
- [25] D. Zagier, *Ramanujan’s mock theta functions and their applications*, Séminaire Bourbaki, Astérisque **326** (2009), 143–164.
- [26] S. Zweegers, *Mock theta functions*, Ph.D. thesis, Utrecht University (2002).

MATHEMATICAL INSTITUTE, UNIVERSITY OF COLOGNE, WEYERTAL 86-90, 50931 COLOGNE, GERMANY

E-mail address: kbringma@math.uni-koeln.de

E-mail address: bkane@math.uni-koeln.de

E-mail address: szwegers@math.uni-koeln.de